Goal: Given “score-equivalent” data, construct functional harmonic analysis:

\[
\begin{array}{ccc}
\text{I} & \text{IV} & \text{V} \\
\text{C maj} & & \text{F maj}
\end{array}
\]

Applications:

1. Harmony-based music queries

2. automatically generated one-dimensional music representation (similarity measures)
Music can be partitioned in segments have constant key:

\[ k = (t, m) \in \{0, \ldots, 11\} \times \{\text{Major, Minor, \ldots}\} \]

Key has associated scale

Major: 0 1 2 3 4 5 6 7 8 9 10 11 12
Minor: 0 1 2 3 4 5 6 7 8 9 10 11 12

Triad = alternating triple of scale tones

Major Traid:
1 I 0 4 7
2 (ii) 2 5 9
3 (iii) 4 7 11
4 (IV) 5 9 0
: : : :
7 (vii) 11 2 5

**Data:** Music is divided into abstract time intervals (beats, measures, etc.)
Let \( y_n = y_n^1, \ldots, y_n^K \) be pitches in \( \{0, \ldots, 11\} \) in \( n \)th time interval.

**Object:** Find harmonic label for each interval:

\[
\text{\( n \)th label} = x_n \in \left\{0, \ldots, 11\right\} \times \{\text{Major, Minor}\} \times \{1, \ldots, 7\}
\]

\( \text{tonic mode chord} \)
Let $X_1, \ldots, X_N$ be sequence of harmonic labels. Assume $X$ is Markov chain:

$$p(x_n|x_1, \ldots x_{n-1}) = p(x_n|x_{n-1})$$

Is this reasonable?

1. keys tend to persist for long periods of time
2. chords tend to persist, but not as long as keys
3. chords tend to move in familiar patterns: boogie woogie, Pachelbel

Model data $y_n$ as observation of random variable $Y_n$. Model

$$p(y_n|x_1, \ldots, x_n, y_1, \ldots y_{n-1}) = p(y_n|x_n)$$

Is this reasonable?

**Graphical Depiction of Conditional Independencies:**

![Graphical Depiction of Conditional Independencies](image.png)
Model has lots of parameters:

\[
p(\begin{array}{c}
x' \\
12 \times 2 \times 7
\end{array} \mid \begin{array}{c}
x \\
12 \times 2 \times 7
\end{array}) = 28224 \text{ parameters}
\]

\[
p(\begin{array}{c}
y \\
- - -
\end{array} \mid \begin{array}{c}
x \\
12 \times 2 \times 7
\end{array}) = \text{many parameters too}
\]

A complication: Harmonic importance ↔ measure position

For pitches \( y = y^1, \ldots, y^K \) let \( r = r^1, \ldots, r^K \) describe the measure positions:

\[ r^k \in \{0, \ldots, 5\} \]

where lower values mean stronger measure positions.

Output probs modeled as \( p(y|x, r) \).

\[
p(y|x, r) = p(y^1, \ldots, y^K \mid x, r^1, \ldots, r^K)
\]

\[
= \prod_{k=1}^{K} p(y^k \mid x, r^k)
\]

\[
= \prod_{k=1}^{K} p(y^k \mid t, m, c, r^k)
\]

\[
= \prod_{k=1}^{K} p(y^k - t \mid m, c, r^k)
\]

\[
= \prod_{k=1}^{K} p(F(y^k - t, m, c) \mid r^k)
\]

\[
p(x' \mid x) = p(t', m', c' \mid t, m, c)
\]

\[
= p(t', m' \mid t, m) p(c' \mid t', m', t, m, c)
\]

\[
= p(t' - t, m' \mid m) \begin{cases} p(c' \mid c) & \text{if } t' = t, m' = m \\ p(c') & \text{otherwise} \end{cases}
\]
Forward Probabilities: $\alpha_n(x_n) = p(x_n, y_1, \ldots, y_n)$

\[
\begin{align*}
\alpha_{n+1}(x_{n+1}) &= p(x_{n+1}, y_1, \ldots, y_{n+1}) \\
&= \sum_{x_n} p(x_n, x_{n+1}, y_1, \ldots, y_{n+1}) \\
&= \sum_{x_n} p(x_n, y_1, \ldots, y_n)p(x_{n+1}|x_n)p(y_{n+1}|x_{n+1}) \\
&= \sum_{x_n} \alpha_n(x_n)p(x_{n+1}|x_n)p(y_{n+1}|x_{n+1})
\end{align*}
\]

Backward Probabilities: $\beta_n(x_n) = p(x_n, y_{n+1}, \ldots, y_N)$ computed analogously. Then

\[
p(x_n|y_1, \ldots, y_N) = \frac{\alpha(x_n)\beta(x_n)}{\sum_{x_n'} \alpha(x_n')\beta(x_n')}
\]

Could reestimate $p(y|x)$, for example, by

\[
\hat{p}(y|x) = \frac{\sum_{y_n=y} p(x_n = x|y_1, \ldots, y_N)}{\sum p(x_n = x|y_1, \ldots, y_N)}
\]

Forward-Backward algorithm alternates between computing $\alpha$ and $\beta$ and reestimating output and transition probabilities.
Seek
\[
\arg \max_x p(x|y) = \arg \max_x p(x, y)
\]
\[
p(x, y) = p(x_1, \ldots, x_N, y_1, \ldots, y_N)
\] factors as
\[
p(x, y) = p(x)p(y|x)
= p(x_1)p(x_2|x_1) \cdots p(x_N|x_{N-1})p(y_1|x_1) \cdots p(y_n|x_n)
= p(x_1)p(y_1|x_1)p(x_2|x_1)p(y_2|x_2) \cdots p(x_N|x_{N-1})p(y_N|x_N)
\]
1. Haydn Piano Sonata
2. Harder: Chopin “Raindrop Prelude”
3. Still Harder: Debussy Prelude (Suite Bergamasque)

**More sophisticated choice:** Model $p(y_n|x_ny_{n-1})$